SOLUTION OF A NONSTEADY TRANSPORT PROBLEM BY THE FLUX METHOD

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A system of flux equations is proposed that is completely equivalent to the P_1 approximation, and has a more rigorous solution to the nonsteady transport problem than does the diffusion equation. An example using the proposed method is presented.

It is known that the one-velocity Boltzmann equation has an exact solution only for an infinite medium, and can be solved for a semiinfinite medium (the Milne problem) [1]. For a bounded medium, a solution can be obtained by expanding in an infinite series of Legendre polynomials. Limiting ourselves to the first two terms of the expansion, we obtain the equation of the so-called P_1 approximation [2]:

$$\frac{1}{\bar{v}}\frac{\partial N_0}{\partial t}+\frac{1}{3}\frac{\partial N_1}{\partial x}+\frac{N_0}{l_a}=\frac{S_0}{\bar{v}}, \tag{1}$$

$$\frac{1}{\bar{v}}\frac{\partial N_1}{\partial t}+\frac{1}{3}\frac{\partial N_0}{\partial x}+\frac{N_1}{l}=0. \tag{2}$$

From (1) and (2), by simple transformations, we can obtain a single equation for the total concentration:

$$\frac{3D^*}{\bar{v}^2}\frac{\partial^2 N}{\partial t^2}+\left(1+\frac{l}{l_a}\right)\frac{\partial N}{\partial t}=D^*\frac{\partial^2 N}{\partial t^2}-\frac{N}{\tau}+S+\frac{l}{\bar{v}}\frac{\partial S}{\partial t}, \tag{3}$$

where $D^* = \bar{v}l/3$ is the diffusion constant; τ is the lifetime; $\bar{v}$ is the mean thermal velocity; l is the mean free path for an interaction, where

$$\frac{1}{l}=\frac{1}{l_a}+\frac{1}{l_s};$$

l_a is the mean free path for absorption; l_s is the mean free path for scattering; and S is a source function.

Introducing the simplifying assumptions

$$\frac{\partial^2 N}{\partial t^2}=0,\quad \frac{l}{l_a}\ll 1,\quad \frac{\partial S}{\partial t}=0, \tag{4}$$

we obtain the well-known diffusion equation [3]

$$D_0\frac{\partial^2 N}{\partial x^2}-\frac{N-N_0}{\tau}=\frac{\partial N}{\partial t}, \tag{5}$$

where

$$D_0=\frac{\bar{v}l_s}{3},\quad N_0=\tau S_0,$$

which is also commonly used for the solution of all transport problems in bounded media, including nonsteady problems [4]. Since we neglect the second time derivative, the diffusion equation becomes an equation of long-range interaction.

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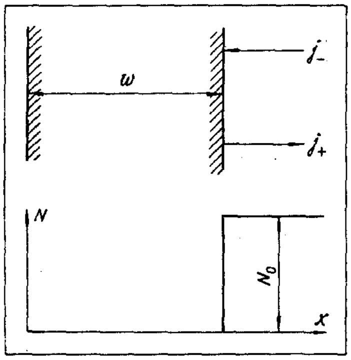

Fig. 1. Flux diagram in a plane layer for t = 0.

As a result of this situation, in order to solve nonsteady problems it turns out that the formation of a concentration discontinuity in the diffusion particles at some point on the x axis leads to a simultaneous formation of a certain concentration value at every point of the axis being considered. Furthermore, the results obtained by taking account of the action of a transient perturbation (a δ pulse) diverge for $t \to 0$.

Meanwhile, there has been no need to introduce the simplifications (4). We show that from Eqs. (1) and (2) we can obtain the flux equations, which not only have the convenience of the diffusion equation (5), but also are considerably more flexible in the introduction of various boundary conditions.

We introduce the fluxes j_+ and j_-, moving in the positive and negative directions along the x axis [5]:

$$j_+ = \frac{4\pi\bar{v}N_0}{4} + \frac{4\pi\bar{v}N_1}{6}, \tag{6}$$

$$j_- = \frac{4\pi\bar{v}N_0}{4} - \frac{4\pi\bar{v}N_1}{6}. \tag{7}$$

From (1) and (2), taking account of (6) and (7), we now obtain

$$\frac{7}{4\bar{v}}\frac{\partial j_+}{\partial t} + \frac{1}{4\bar{v}}\frac{\partial j_-}{\partial t} + \frac{\partial j_+}{\partial x} = \frac{3}{4l}(j_+ - j_-) - \frac{j_+ + j_-}{l_a} + 2\pi S_0, \tag{8}$$

$$-\frac{1}{4\bar{v}}\frac{\partial j_+}{\partial t} - \frac{7}{4\bar{v}}\frac{\partial j_-}{\partial t} + \frac{\partial j_-}{\partial x} = \frac{3}{4l}(j_+ - j_-) + \frac{j_+ + j_-}{l_a} - 2\pi S_0. \tag{9}$$

For the problem with zero initial conditions, we can, using the Laplace transformation, write Eqs. (8) and (9) in the form

$$\frac{\partial j_+}{\partial x} = -\frac{3}{4l}(j_+ - j_-) - \frac{j_+ + j_-}{l_a} - \frac{7}{4\bar{v}}sj_+ - \frac{1}{4\bar{v}}sj_- + 2\pi S_0, \tag{10}$$

$$\frac{\partial j_-}{\partial x} = -\frac{3}{4l}(j_+ - j_-) + \frac{j_+ + j_-}{l_a} + \frac{7}{4\bar{v}}sj_+ + \frac{1}{4\bar{v}}sj_- - 2\pi S_0, \tag{11}$$

where s is the Laplace operator. Finally, Eqs. (10) and (11), in terms of the flux method, can be represented as

$$\frac{dj_+}{dx} = -\left[k - \frac{1}{8}m(s)\right](j_+ - j_-) - m(s)\,j_+, \tag{12}$$

$$\frac{dj_-}{dx} = -\left[k - \frac{1}{8}m(s)\right](j_+ - j_-) + m(s)\,j_-, \tag{13}$$

where

$$k = \frac{3}{4l_s}, \quad m(s) = \frac{2}{\bar{v}}\left(\frac{\bar{v}}{l_a} + s\right), \quad m = \frac{2}{l_a}. \tag{14}$$

The system of equations (12)-(13) has the general solution

$$j_+(x, s) = A(s)R_\infty(s)\exp[q(s)x] + B(s)\exp[-q(s)x], \tag{15}$$

$$j_-(x, s) = A(s)\exp[q(s)x] + B(s)R_\infty(s)\exp[-q(s)x], \tag{16}$$

where

$$q(s) = \frac{\sqrt{3}}{\bar{v}}\sqrt{\left(s + \frac{\bar{v}}{l_a}\right)\left(s + \frac{\bar{v}}{l}\right)}, \tag{17}$$

$$R_\infty(s) = \frac{q(s) - m(s)}{q(s) + m(s)}, \tag{18}$$

A(s) and B(s) are integration constant determined from the boundary conditions.*

We consider an example of solving the nonsteady problem by using Eqs. (15) and (16). We assume that on the right boundary of a layer of arbitrary thickness w (Fig. 1) there occurs a discontinuity in concentration N_0 such that

$$N(s) = \frac{N_0}{s}. \tag{19}$$

If the region to the left of the layer being considered is assumed to be nonreflecting, then in this case the boundary conditions are of the form

$$j_+(0, s) = 0, \tag{20}$$

$$j_+(w, s) + j_-(w, s) = \frac{N_0}{s}\frac{\bar{v}}{2}. \tag{21}$$

From (16) and (21) we obtain

$$B(s) = -A(s) R_\infty(s). \tag{22}$$

Equation (23) follows from (15), (16), and (21):

$$A(s) = \frac{N_0\bar{v}}{2s} \frac{1}{[1 + R_\infty(s)]\{\exp[q(s)w] - R_\infty(s)\exp[-q(s)w]\}}. \tag{23}$$

Then the transform of the flux emerging from the layer will be of the form

$$J_-(0, s) = A(s)[1 - R_\infty^2(s)] = \frac{\frac{N_0\bar{v}}{2s}[1 - R_\infty(s)]}{\exp[q(s)w] - R_\infty(s)\exp[-q(s)w]} \tag{24}$$

Using the notation given above, and carrying out simple transformations, we finally obtain

$$J_-(0, s) = \frac{N_0\bar{v}}{2s}\frac{m(s)}{q(s)} \frac{1}{\mathrm{Sh}[q(s)w] + \frac{m(s)}{q(s)}\mathrm{Ch}[q(s)w]} \tag{25}$$

Obtaining the inverse transform of the transform (25) is difficult in general; however, for certain limiting cases, it becomes rather simple. Since for $s \to \infty$

$$\lim J_-(0, s)_{s\to\infty} = \frac{N_0\bar{v}}{2s}\exp\left[-\frac{2w}{\bar{v}}s\right], \tag{26}$$

the initial behavior of the flux emerging from the layer can be described by the equations

$$J_-(0) = 0 \quad \text{for} \quad t < \frac{2w}{\bar{v}}, \tag{27}$$

$$J_-(0) = \frac{N_0\bar{v}}{2} \quad \text{for} \quad t = \frac{2w}{\bar{v}}. \tag{28}$$

From (25) we can also easily determine the steady-state value

$$\lim J_-(0, t)_{t\to\infty} = \lim[J_-(0, s)s]_{s\to\infty} = \frac{N_0\frac{\bar{v}m}{2q}}{\mathrm{Sh}\,qw + \frac{m}{q}\mathrm{Ch}\,qw}. \tag{29}$$

In order to analyze the final stage of the transient process and to determine the time for it to be established, we can use the following approximations. As usual, we can assume that $\bar{v}/l \gg s$; we then have

*The flux equations for the steady-state case were first obtained by Shockley [6].

$$q(s) = q_0 \sqrt{1 + s\tau}\,, \tag{30}$$

$$\frac{m(s)}{q(s)} = \frac{m_0}{q_0}\sqrt{1 + s\tau}\,, \tag{31}$$

where

$$q_0 = \sqrt{2km_0 + \frac{3}{4} m_0^2}\,. \tag{32}$$

From (31) and (32) it is easy to establish that for $l_a \gg l_s$ and $s \to 0$ we have

$$\frac{m(s)}{q(s)} \ll 1. \tag{33}$$

Taking account of all these simplifications, we can write Eq. (25) in the form

$$J_-(0, s) = \frac{N_0\bar{v}}{2s}\,\frac{m_0}{q_0}\,\frac{\sqrt{1 + s\tau}}{\mathrm{Sh}\,[q_0 w \sqrt{1 + s\tau}\,]}\,. \tag{34}$$

The inverse transform of (34) can easily be obtained for the limiting cases of small and large thickness w. If q(s)w ≪ 1, then

$$J_-(0, t) = \mathrm{const} = \frac{N_0 D^*}{w}\,. \tag{35}$$

For large thickness, when q(s)w ≫ 1, Eq. (34) implies

$$J_-(0, t) = \frac{N_0\bar{v}}{2}\,\frac{m_0}{q_0}\left\{\exp[-q_0 w]\,\mathrm{erfc}\left(\frac{q_0 w}{2}\sqrt{\frac{\tau}{t}} - \sqrt{\frac{t}{\tau}}\right)\right.$$
$$\left. - \exp[q_0 w]\,\mathrm{erfc}\left(\frac{q_0 w}{2}\sqrt{\frac{\tau}{t}} - \sqrt{\frac{t}{\tau}}\right) + \frac{2}{\sqrt{\pi\frac{t}{\tau}}}\exp\left[-\frac{t}{\tau} - \frac{q_0^2 w^2}{4}\,\frac{\tau}{t}\right]\right\}. \tag{36}$$

Thus the flux method enables us to determine the initial and steady-state values of the flux emerging from a layer of finite thickness and also to describe the process of its establishment.

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